Biased random walks on complex networks: the role of local navigation rules

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We study the biased random walk process in random uncorrelated networks with arbitrary degree distributions. In our model, the bias is defined by the preferential transition probability, which, in recent years, has been commonly used to study efficiency of different routing protocols in communication networks. We derive exact expressions for the stationary occupation probability, and for the mean transit time between two nodes. The effect of the cyclic search on transit times is also explored. Results presented in this paper give the basis for theoretical treatment of the transport-related problems on complex networks, including quantitative estimation of the critical value of the packet generation rate.

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The problem of wandering at random in a network (or lattice) finds applications in virtually all sciences [1, 2]. With only minor adjustments random walks may represent thermal motion of electrons in a metal, or migration of holes in a semiconductor. The continuum limit of the random walk model is known as diffusion. It may describe Brownian motion of a particle immersed in a fluid, as well as heat propagation, bacterial motion, and even fluctuations in the stock market. Recently, the concept of random walks has been also applied to explore traffic in complex networks. The spectrum of network related problems include, among many others, ordinary traffic in a city, distribution of goods and wealth in economies, biochemical and gene expression pathways, and finally search (or routing) strategies in the Internet and other communication networks [3, 4, 5, 6, 7, 8, 9].

In this paper, we deal with biased random walks on complex networks, and we explore the role of different local navigation rules on the mean first-passage (or transit) time between any pair of nodes [10]. The biased random walk model defined on scale-free networks is particularly interesting since it has been considered as a mechanism of transport and search in real networks, including the Internet. For a long time one has believed that the most optimum transport-related processes are based on shortest paths between two nodes under consideration. At the moment, one has however understood that such a routing strategy would require a global knowledge on network topology, which is often not available. Moreover, one can simply imagine that routing strategies based on shortest paths may create inconvenient queue congestions in scale-free networks, given that the majority of the shortest paths pass through hub nodes in such structures. It has been also realized that a possible alternative is to consider local navigation rules instead of global knowledge. As a consequence, a number of adequate models have been proposed (see e.g. [5, 8]). In general, the models mimic traffic in complex networks by introducing packets (particles) generation rate, as well as assigning a randomly selected source and a random destination to each packet. In these models, a common observation is that the traffic exhibits continuous phase transition from free flow to the congested phase as a function of the packet generation rate. In the free flow state, the numbers of created and delivered particles are balanced, while in the jammed state, the number of packets accumulated in the network increases with time. In this paper we show that the random walk model, although very simple, correctly describes properties of the proposed traffic models in the free flow state. We calculate transit times characterizing this state. We also give some suggestions how to calculate the critical packet generation rate.

Technically, we define our random walks as follows. We consider random uncorrelated networks with given node degree distributions $P(k)$ [11]. The networks are also known as random graphs or configuration model, and they have been repeatedly shown to be very useful in modelling different phenomena taking place on networks. We assume that the networks are connected, i.e. there exists a path between each pair of nodes. Given the graph structure, the diffusing particle (packet) is created at a randomly selected node, and it is assigned a random destination node. In the next time steps the particle passes from a node to one of its neighbors being directed by local navigation rules. In practice, it means that being in a certain node i random walker performs a local search in its neighborhood (up to the first, second, or further orders) looking if the destination node is within the search area. If the destination is found, the particle is delivered directly to the target following the shortest path (the rule is known as the cyclic search [5]). Otherwise, the particle continues biased random walk, i.e. the next position (a node j) is chosen according to the prescribed probability w_{ij}.

In the following, to explore transit times characterizing biased random walks in uncorrelated networks with arbitrary degree distributions $P(k)$, we partially reproduce and generalize standard calculations for the mean

first passage time in periodic lattices [12]. At the beginning, we work out some general concepts related to biased random walks without the cyclic search. In particular, we calculate the stationary occupation probability P_i^∞ for the diffusing particle, which describes the probability that the particle is located at the node i in the infinite time limit. Then, performing simple textbook calculations we derive formulas for the mean transit time between any pair of nodes (we would like to stress that some time ago similar calculations were done for unbiased random walks on complex networks [13]; results presented in our paper encompass the results of Ref. [13] as a special case). The role of the cyclic search on transit times is explored via a simple renormalization trick applied to nodes' degrees.

Thus, let us consider a particle that hops at discrete times between neighboring nodes of a random network described by the adjacency matrix $\mathbf{A}$. Let $P_{ij}(t)$ be the probability that the particle starting at site i at time $t = 0$ is at site j at time t. The evolution of this occupation probability is given by the master equation

$$P_{ij}(t+1) = \sum_{l=1}^{N} A_{lj} w_{lj} P_{il}(t), \tag{1}$$

where the meaning of w_{lj} was already exposed, whereas A_{lj} represents element of the adjacency matrix, which is equal to 1 if there exists a link between l and j, and 0 otherwise. In the rest of the paper we perform a detailed analysis of the local navigation rules defined by the preferential transition probability [8, 14]

$$w_{lj} = \frac{k_j^\alpha}{\sum_{m=1}^{k_l} k_m^\alpha}, \tag{2}$$

where the sum in the denominator runs over the neighbors of the node l, and the exponent α is the model free parameter. Note that according to the formula (2) the transition probability from l to j in our biased random walk depends only on the connectivity of the next-step node j. Note also that for $\alpha = 0$ we recover the ordinary unbiased random walk studied by Noh and Rieger [13].

In order to calculate the stationary occupation probability P_i^∞ characterizing the studied biased random walks we average the master equation (1) over the ensemble of the considered networks (i.e. we apply mean field approximation to this equation)

$$P_j^\infty \simeq \sum_{l=1}^{N} \langle A_{lj} \rangle \langle w_{lj} \rangle P_l^\infty. \tag{3}$$

Now, before we proceed further, let us recall a few structural properties of uncorrelated networks with a given node degree distribution. First, one can show that probability of a link between any pair of nodes l and j with degrees respectively equal to k_l and k_j is given by (see Eq. (27) in [15])

$$\langle A_{lj} \rangle = \frac{k_l k_j}{\langle k \rangle N}. \tag{4}$$

Second, since in uncorrelated networks the node degree distribution $Q(k_m/k_l)$ of the nearest neighbors of a node l does not depend on k_l (compare Eq. (1) in [16], and Eq. (4) in [17])

$$Q(k_m/k_l) = \frac{k_m}{\langle k \rangle} P(k_m), \tag{5}$$

the normalization factor in the formula (2) is equal to

$$\sum_{m=1}^{k_l} k_m^\alpha = k_l \sum_{m=1}^{k_l} k_m^\alpha Q(k_m/k_l) = \frac{\langle k^{\alpha+1} \rangle}{\langle k \rangle} k_l, \tag{6}$$

and the transition probability w_{lj} between l and j averaged over different network realizations may be written as

$$\langle w_{lj} \rangle = \frac{\langle k \rangle}{\langle k^{\alpha+1} \rangle k_l} k_j^\alpha. \tag{7}$$

Finally, inserting the relations (4) and (7) into the simplified master equation (3), after some algebra, one obtains

$$P_j^\infty = \frac{k_j^{\alpha+1}}{N \langle k^{\alpha+1} \rangle}. \tag{8}$$

Note that for $\alpha = 0$, which stands for the unbiased random walk, the stationary distribution is, up to normalization, equal to the degree of the the node j, i.e. $P_j^\infty \sim k_j$. It means that the more links a node has, the more often it will be visited by a random walker. Note also that for $\alpha = -1$, which represents the anti-preferential transition probability $w_{lj} \sim 1/k_j$, the stationary occupation probability is uniform $P_j^\infty = 1/N$.

To test the validity of Eq. (8) we have numerically calculated the fraction of random walkers found in nodes with a given connectivity k_j. The expected power law relation $P_j^\infty \sim k_j^{\alpha+1}$ was found in all the considered α-cases, and for different classes of the analyzed networks (i.e. classical random graphs, and scale-free networks $P(k) \sim k^{-\gamma}$ with the characteristic exponent $\gamma = 3$), see Fig. 1. The same scaling behavior was found in Ref. [8] for the number of packets moving simultaneously on BA networks [18] in the free flow state. In the mentioned paper, a packet routing strategy based on the preferential transition probability (2), and the so-called path iteration avoidance, which means that no link can be visited twice by the same packet, has been considered. At each time step R packets have been generated in the network, and a fixed node capacity C, that is the number of packets a node can forward to other nodes, has been assumed. The fact that our results coincide with those of Wang et al. [8]

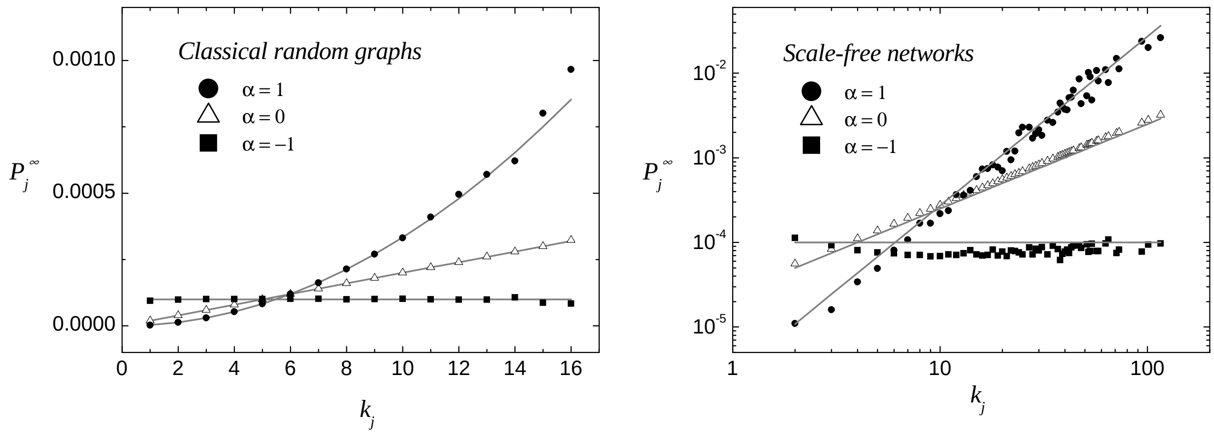

FIG. 1: Stationary probability distributions $P_j^\infty(k)$ calculated for different values of the parameter α in classical random graphs and scale-free networks. Solid lines correspond to the theoretical prediction of Eq. (8). All numerical simulations have been done for networks of size $N = 10^4$ and averaged over 10^4 random walkers. In the case of classical random graphs $\langle k \rangle = 5$ was assumed, whereas in scale-free networks $\gamma = 3$ and $k_{min} = 2$ were chosen.

shows that packets may be considered as non-interacting particles (i.e. independent biased random walkers) in the free flow, stationary state. One can also show that the approach may be used to estimate the critical value of packets generation rate R_c [19], as the parameter should fulfill a kind of balance equation between the node's processing efficiency C, and the number of delivered packets delivered $P_j^\infty R_c \langle T_{ij} \rangle$, where $\langle T_{ij} \rangle$ stands for the mean first-passage time (13) averaged over all pair of nodes, and respectively $R_c \langle T_{ij} \rangle$ corresponds to the total number of packets distributed over the whole network.

The first-passage probability $F_{ij}(t)$, namely the probability that the random walk starting at the node i visits j for the first time at time t satisfies the well-known convolution relation [10, 13]

$$P_{ij}(t) = \delta_{t0}\delta_{ij} + \sum_{\tau=0}^{t} P_{jj}(t-\tau) F_{ij}(\tau). \tag{9}$$

The delta function term in the last equation (9) accounts for the initial condition that the walk starts at $i = j$. Applying the Laplace transform, defined as $\widetilde{f}(s) = \sum_{t=0}^{\infty} e^{-st} f(t)$, to this equation leads to the fundamental expression

$$\widetilde{F}_{ij}(s) = \frac{\widetilde{P}_{ij}(s) - \delta_{ij}}{\widetilde{P}_{jj}(s)}, \tag{10}$$

in which the Laplace transform of the first-passage probability $\widetilde{F}_{ij}(s)$ is determined by the corresponding transform of the probability distribution $\widetilde{P}_{ij}(s)$. Consequently, due to the fact that all moments

$$R_{ij}^{(n)} = \sum_{t=0}^{\infty} t^n \left(P_{ij}(t) - P_j^\infty \right) \tag{11}$$

of the exponentially decaying part of $P_{ij}(t)$ are finite, expanding $\widetilde{P}_{ij}(s)$ as a power series in s

$$\widetilde{P}_{ij}(s) = \frac{P_j^\infty}{1 - e^{-s}} + \sum_{n=0}^{\infty} (-1)^n R_{ij}^{(n)} \frac{s^n}{n!}, \tag{12}$$

and then inserting (12) into (10), and again expanding the result as a series in s, one finally obtains the formula for the mean transit time between i and j

$$\begin{aligned} T_{ij} &= \sum_{t=0}^{\infty} t F_{ij}(t) = -\widetilde{F}'_{ij}(0) \\ &= \begin{cases} 1/P_j^\infty, & \text{for } j = i \\ \left[R_{jj}^{(0)} - R_{ij}^{(0)} \right] / P_j^\infty, & \text{for } j \neq i \end{cases} . \end{aligned} \tag{13}$$

At the moment, let us remind that P_j^∞ (8) corresponds to the stationary occupation probability, which has been already calculated.

Figure 2 shows how the mean first return time T_{ii} of the biased diffusing particle wandering in random network depends on k_i. In the figure, numerically calculated transit times are indicated by scattered points, whereas their values predicted by the theory (13), namely

$$T_{ii} = \frac{N \langle k^{\alpha+1} \rangle}{k_i^{\alpha+1}}, \tag{14}$$

are represented by solid lines. Subsets given in the figure show how the mean first return time $\langle T_{ii} \rangle$ averaged over all nodes depends on α (i.e. on local navigation rules governing the diffusing particle)

$$\langle T_{ii} \rangle = N \langle k^{\alpha+1} \rangle \langle k^{-\alpha-1} \rangle, \tag{15}$$

and they indirectly show how fast the biased random walk is. The minimum value of $\langle T_{ii} \rangle$ observed for $\alpha_m \simeq$

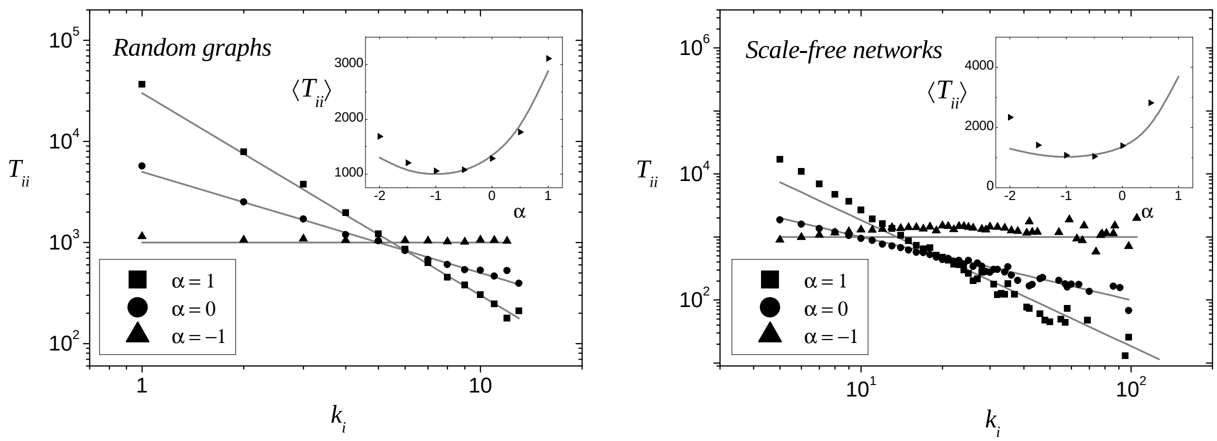

FIG. 2: Mean first return time T_{ii} vs node degree k_i (main panels), and $\langle T_{ii} \rangle$ vs α (insets) in classical random graphs and scale-free networks. Numerical calculations have been done for networks of size $N = 10^3$. $\langle k \rangle = 5$ and $\langle k \rangle = 2k_{min} = 10$ have been assumed in classical random graphs and scale-free networks (with $\gamma = 3$), respectively. Data presented in the figure have been averaged over 10^3 random walkers running in 10^2 different network configurations.

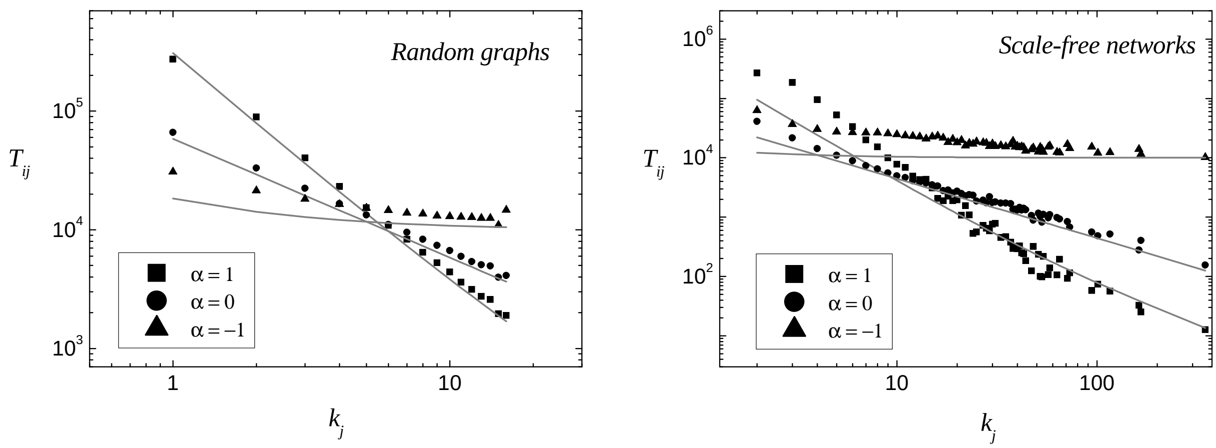

FIG. 3: Mean transit time T_{ij} between two nodes i and j vs connectivity of the target node k_j in classical random graphs and scale-free networks. Numerical calculations have been done for networks of size $N = 10^4$. $\langle k \rangle = 5$ and $\langle k \rangle = 2k_{min} = 4$ have been assumed in classical random graphs and scale-free networks (with $\gamma = 3$), respectively. Data presented in the figure have been averaged over 10^2 random walkers running in 50 different network configurations.

-1 in classical random graphs indicates that the anti-preferential transition probability (2) causes the slowest exploration of the considered networks, which, in turn, causes that in the case of such a navigation rule the relaxation part of the occupation probability $P_{ii}(t) - P_i^\infty$ converges to zero much slower then in the case of other values of the parameter α (the same reasoning applies to the case of $\alpha_m \simeq -0.5$ in scale-free networks). The reasoning implies that although, in general, the parameters $R_{jj}^{(0)}$ and $R_{ij}^{(0)}$ in the formula (13) for the mean transit time T_{ij} can not be easily calculated, the expected for $|\alpha - \alpha_m| \neq 0$ fast convergence of the occupation probability $P_{ij}(t)$ towards the stationary distribution P_j^∞ allows one to simplify the relation

$$T_{ij} \simeq \left(\sum_{t=0}^{2} (P_{jj}(t) - P_j^\infty) - \sum_{t=0}^{0} (P_{ij}(t) - P_j^\infty) \right) / P_j^\infty = \frac{N\langle k^{\alpha+1}\rangle}{k_j^{\alpha+1}} + \frac{N\langle k\rangle^2}{\langle k^2\rangle} \frac{1}{k_j} - 2. \quad (16)$$

In figure 3 one can see that the theoretical prediction of the last equation (16) quite good agrees with numerical calculations of T_{ij}. As expected, the approximate formula (16) works better for the parameter $\alpha > \alpha_m$. We have also checked that the mean first passage time T_{ij} between any pair of nodes does not depend on the source node i in the considered networks. It only depends on the destination node j.

Knowing the mean transit time (13) of the biased ran-

dom walk, the role of the cyclic search on the quantity can be calculated through simple renormalization trick applied to nodes' degrees. The trick consists in dividing the walk between any pair of nodes i and j into two parts. The first part, before the diffusing particle hits neighborhood of the target, and the second part, when the particle sees its destination, and follows the shortest path. Distinguishing between the two parts allows to treat the first part as an ordinary biased random walk from a node i to the node J with the renormalized degree k_J equal to

$$k_J = k_j \left(\frac{\langle k^2 \rangle}{\langle k \rangle} - 1 \right)^x , \tag{17}$$

where $\langle k^2 \rangle / \langle k \rangle$ represents the average connectivity of the nearest neighbors in uncorrelated networks, and x is the parameter describing the depth of the search area. Now, since the mean transit time of the second part of the walk is equal to x the mean first passage time characterizing the whole walk between i and j (i.e. its both parts) is given by the sum

$$T_{ij}^{(x)} = x + T_{iJ} = x + \left[R_{JJ}^{(0)} - R_{iJ}^{(0)} \right] / P_J^{\infty}, \tag{18}$$

where the quantities T_{iJ}, P_J^{∞}, $R_{JJ}^{(0)}$, and $R_{iJ}^{(0)}$ apply to the network, in which the original target node j together with its nearest neighborhood was replaced by a single node J of degree k_J (17). In this case, however, due to difficulties related to the precise calculation, or even estimation of the parameters $R_{jj}^{(0)}$ and $R_{ij}^{(0)}$, the direct verification of the validity of the formula (18) is impossible. However, we have numerically checked that the mean first passage time $T_{ij}^{(x)}$ characterizing the cyclic search does not depend is the source node i, and in the first approximation it is proportional to $k_J^{-\alpha-1}$ (compare Eq. (16)).

In summary, we have studied the biased random walk process in random uncorrelated networks with arbitrary degree distributions. In our model, the bias was defined by the preferential transition probability (2) (see also another paper on biased diffusion in random networks [20]). We have calculated the expression for the stationary occupation probability, and we have derived formulas for the mean first passage times between any pair of nodes. The role of the cyclic search on transit times was explored via a simple renormalization trick applied to nodes' degrees. We have also shown that the random walk approach can be used to explain some properties of traffic dynamics in communication networks. Other traffic-related problems, that can be solved using the approach include, among many others, microscopic explanation of the phase transition from free flow to the jammed phase, and quantitative estimation of the critical value of the packet generation rate in scale-free networks [8]. We leave the problems to our future work [19].

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